· Chain rule $\rightarrow$ $\nabla f(a) \perp S$, $a \in S = f^{-1}(c)$
$\hookrightarrow$ Implicit differentiation.

· Extreme value theorem

$f: A\ (\subseteq \mathbb{R}^n) \longrightarrow \mathbb{R}$ $\Rightarrow$ f has global max/min.

continuous ↑ closed & bounded

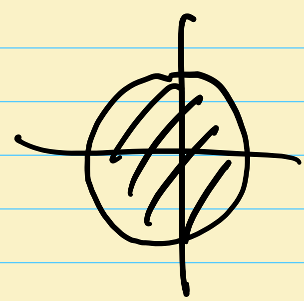
· $a \in \mathrm{int}(A)$ local extremum $\Rightarrow$ $\nabla f(a) = 0$.

Finding extrema

$a \in A$ extrema $\rightarrow$ $a \in \partial A$ $\Rightarrow$ study directly
$\hookrightarrow$ $a \in \mathrm{int} A$ $\Rightarrow$ $\nabla f(a) = 0$ or $\nabla f(a)$ DNE

① Find critical points of f in $\mathrm{int}(A)$
② Study f on ∂A : Find max/min of f on ∂A.
③ Compare ① & ②

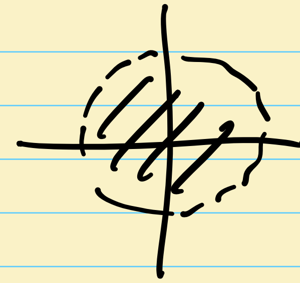
eg Find global max/min of $f(x,y) = x^2 + 2y^2 - x + 3$
define on $x^2 + y^2 \leq 1$



Rmk Domain of $f = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ is closed and bounded.
Also, f is a polynomial, hence f is continuous

$\therefore$ By EVT f has global max/min on A.

(sol) <u>step1</u> study critical points in $\text{int}(A)$



$\text{int}(A) = \{ (x,y) \in \mathbb{R}^2 \mid x^2+y^2<1 \}$

$\nabla f = (2x-1, 4y)$

$$\nabla f = 0 \iff \begin{cases} 2x-1=0 \\ 4y=0 \end{cases} \iff \begin{cases} x=\frac{1}{2} \\ y=0 \end{cases}$$

Note that $(\frac{1}{2}, 0) \in \text{int}(A)$

$\therefore f$ has only one critical point $(\frac{1}{2}, 0)$ in $\text{int}(A)$

$$f(\tfrac{1}{2},0) = \left(\tfrac{1}{2}\right)^2 + 2\cdot 0^2 - \left(\tfrac{1}{2}\right) + 3$$
$$= \tfrac{1}{4} - \tfrac{1}{2} + 3 = \tfrac{11}{4}$$

<u>step2</u> Study f on $\partial A = \{ (x,y) \in \mathbb{R}^2 \mid x^2+y^2=1 \}$

We can parametrize ∂A by $(x,y) = (\cos\theta, \sin\theta)$

for $\theta \in [0, 2\pi]$.

$$f(x,y)\Big|_{\text{on } \partial A} = f(\cos\theta, \sin\theta)$$
$$= \cos^2\theta + 2\sin^2\theta - \cos\theta + 3$$
$$= \cos^2\theta + 2(1-\cos^2\theta) - \cos\theta + 3$$

$$= -\cos^2\theta - \cos\theta + 5$$

$$= -\left(\cos\theta + \frac{1}{2}\right)^2 + \frac{21}{4}$$

$0 \leq \theta \leq 2\pi$.

So $f(\cos\theta, \sin\theta)$ has the maximum value $\frac{21}{4}$

when $\cos\theta = -\frac{1}{2}$ i.e. $(x, y) = \left(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$

min value of $f(\cos\theta, \sin\theta)$ $-\left(1+\frac{1}{2}\right)^2 + \frac{21}{4} = 3$

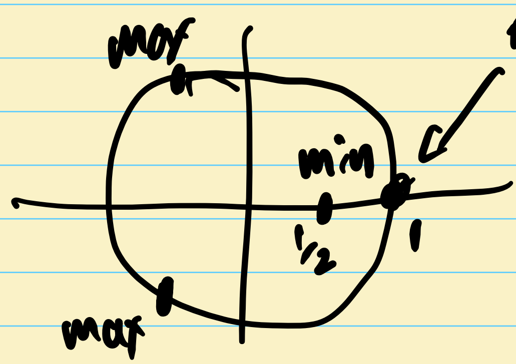
when $\cos\theta = 1$ i.e. $(x, y) = (1, 0)$.

Step 3 Compare step 1 & step 2.

$f\left(\frac{1}{2}, 0\right) = \frac{11}{4}$ ← min at $\left(\frac{1}{2}, 0\right)$

$f\left(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right) = \frac{21}{4}$ ← max at $\left(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$

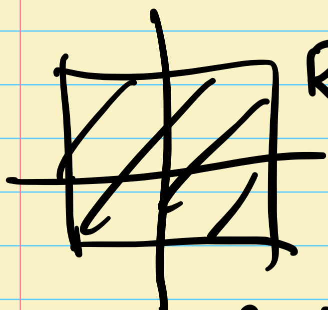
$f(1, 0) = 3$



eg Find global max and min of

$f(x,y) = \sqrt{x^2+y^4} - y$ on

$R = \{(x,y) \in \mathbb{R}^2 \mid -1 \le x, y \le 1\}$

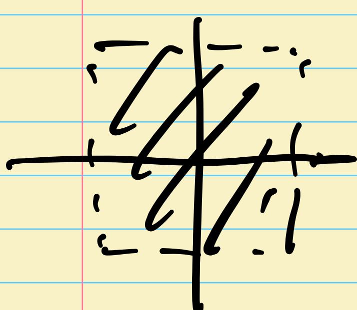
(sol)  R closed and bounded,

f is continuous.

$\therefore$ By EVT, we know that global max/min of f exist on R.

<u>Step 1</u> Study critical points of f in $\operatorname{int}(R)$

$\operatorname{int}(R) = \{-1 < x, y < 1\}$



At $(0,0)$, $\frac{\partial f}{\partial x}(0,0)$ does not exist

$\left(f(x,0) = \sqrt{x^2} = |x| \right)$

$\nabla f(0,0)$ DNE.

If $(x,y) \neq (0,0)$, ∇f exists and

$$\nabla f(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \left(\frac{x}{\sqrt{x^2+y^4}}, \frac{2y^3}{\sqrt{x^2+y^4}} - 1\right)$$

$\therefore \nabla f(x,y) = 0 \iff \begin{cases} \dfrac{x}{\sqrt{x^2+y^4}} = 0 \\ \dfrac{2y^3}{\sqrt{x^2+y^4}} - 1 = 0 \end{cases} \iff \begin{matrix} x = 0 \\ \underbrace{\dfrac{2y^3}{\sqrt{0+y^4}}}_{\checkmark} - 1 = 0 \end{matrix}$

$\rightarrow \dfrac{2y^3}{y^2} - 1 = 0$

$\iff \begin{cases} x = 0 \\ y = \dfrac{1}{2} \end{cases}$

$\therefore$ critical points in $int(A)$

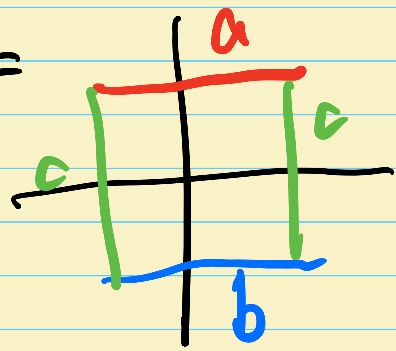
: $(0,0)$ and $(0, \frac{1}{2})$

$\Downarrow$ $\quad\quad\quad\quad\quad\quad$ $\Downarrow$

$f(0,0) = 0$ $\quad\quad\quad\quad$ $f(0, \frac{1}{2}) = -\frac{1}{4}$

<u>Step 2</u> Study f on ∂R

$\partial R =$



On a) $y = 1$, $-1 \leq x \leq 1$.

$f(x,1) = \sqrt{x^2+1} - 1$

$\Rightarrow \quad 0 \leq f \leq \sqrt{2} - 1$

On b) $y = -1$, $-1 \leq x \leq 1$

$f(x,-1) = \sqrt{x^2+1} + 1 \Rightarrow \quad 2 \leq f \leq \sqrt{2} + 1$

On C) $|x|=1$, $-1\le y\le 1$

$$f(x,y)=\sqrt{1+y^4}-y$$

$$\Rightarrow\ 0<f\le\sqrt{2}+1$$

not the sharpest bound, but good enough.

(one-variable calculus ; $\sqrt{2}-1\le f$)

$\therefore$ On ∂R, f has min value 0 at $(0,1)$
max " $\sqrt{2}+1$ at $(\pm 1,-1)$

Step 3 Compare Step 1 & 2

Step 1) $f(0,0)=0$

$f(0,\frac{1}{2})=-\frac{1}{4}$ $\leftarrow$ min

Step 2) $f(0,1)=0$

$f(\pm 1,-1)=\sqrt{2}+1$ $\leftarrow$ max

Finding extrema on an unbounded region.

: EVT not applicable, but it is possible that global extrema exist.

eg Find global extrema of $f(x,y) = x^2+y^2-4x+6y+7$ on $\mathbb{R}^2$.

unbounded.

Rmk Since $f(x,y) = (x-2)^2 + (y+3)^2 - 6$,

f has no global max, global -6 at $(2,-3)$.

strategy for finding global extrema.

: Find a closed and bounded region $R \subset \mathbb{R}^2$

s.t. f is "large enough" outside R.

R & R^c

by EVT, min (& max) of f $\exists$ in R.

f large enough

: the previous steps applicable

(sol) • Find critical points of

$\nabla f = (2x-4,\ 2y+6)$ exists on $\mathbb{R}^2$

$$\nabla f = 0 \iff \begin{cases} 2x-4=0 \\ 2y+6=0 \end{cases} \iff \begin{cases} x=2 \\ y=-3. \end{cases}$$

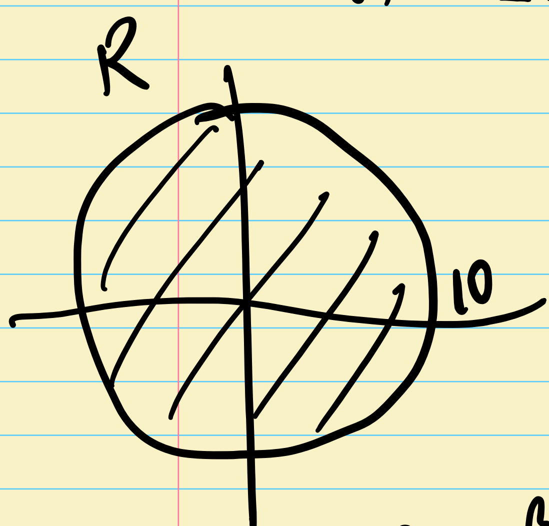
$\therefore$ Only one critical point of f:

$(2,-3)$. $f(2,-3) = -6$.

- observe that

$$f(x,y) = x^2+y^2-4x+6y+7$$
$$= r^2 - 4x+6y+7 \qquad \left(\begin{array}{l} r=\sqrt{x^2+y^2} \\ \geq |x|, |y| \\ \quad x,-x,y,-y \end{array} \right)$$
$$\geq r^2 - 4r - 6r + 7$$
$$= r(r-10)+7.$$

$\therefore$ If $r \geq 10$, $f(x,y) \geq 7 > -6 = f(2,-3)$



Let $R = \overline{B_{10}(0,0)}$

$f(x,y) \geq 7$ outside of R, on ∂R

$\text{int}(R) \ni (2,-3)$

$f(2,-3) = -6.$

$\therefore$ f has no global max

global min -6 at $(2,-3)$.

② $g(x,y) = x^2 - y^2$ has no global max / min

$g(0,y) = -y^2 \Rightarrow$ no global min

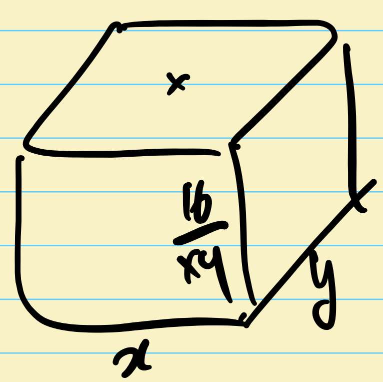
$g(x,0) = x^2 \Rightarrow$ no global max.

Eg Make a box (without top) with volume 16.

Cost : $\left[\begin{array}{l} \$2 \text{ / unit area for the base} \\ \$0.5 \text{ / " sides} \end{array}\right.$

How can we minimize cost?

(Sol)



$C(x,y)$

$$= 2xy + \left(\frac{16}{xy}\cdot x + \frac{16}{xy}\cdot y\right)\cdot 2 \cdot 0.5$$

$$= 2xy + \frac{16}{x} + \frac{16}{y}$$

On the domain $\Omega = \{(x,y) \in \mathbb{R}^2 \mid x > 0, y > 0\}$

Note that Ω is neither closed nor bounded

$\therefore$ EVT is not applicable.

Strategy: Find a region R s.t.

$C >$ (min of of C on R) outside of R

step1 Find critical points

$$C(x,y) = 2xy + \frac{16}{x} + \frac{16}{y}$$

$$\nabla C = \left(2y - \frac{16}{x^2}\ ,\ 2x - \frac{16}{y^2}\right)$$

exists everywhere on Ω.

$$\nabla C = 0 \iff \begin{cases} 2y - \frac{16}{x^2} = 0 \iff y = \frac{8}{x^2} \\ 2x - \frac{16}{y^2} = 0 \end{cases} \qquad 2x - \cancel{16} \cdot \frac{x^4}{\cancel{64}_4} = 0$$

$$\Rightarrow x^3 = 8 \underset{(x>0)}{\Rightarrow} x = 2\ ,\ y = 2.$$

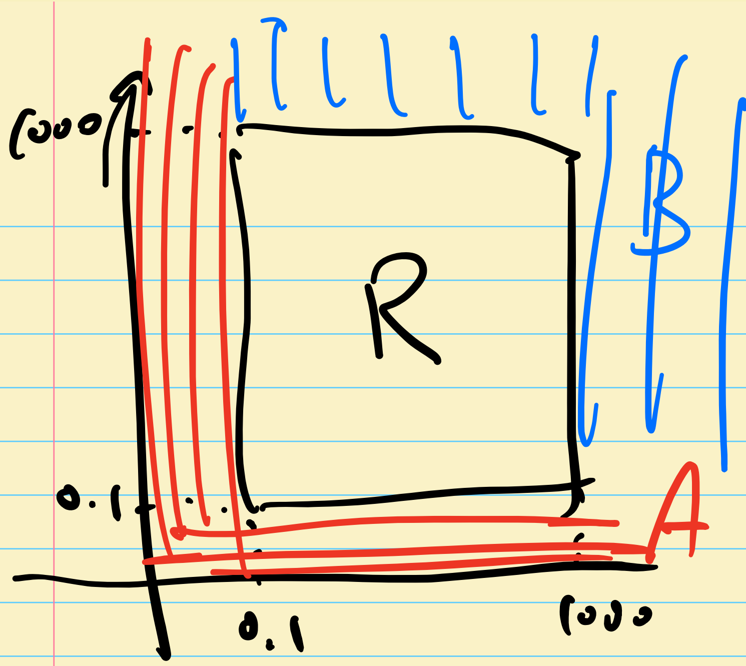
$\therefore$ Only critical point $(2,2)$

$C(2,2) = 24$

step2 Choose R s.t. $C > 24$ on ∂R and outside R.

one possible choice:

$$R = [0.1, 1000] \times [0.1, 1000]$$



(A) $x \le 0.1$ or $y \le 0.1$

$$C(x,y) = 2xy + \frac{16}{x} + \frac{16}{y} > \frac{16}{x} + \frac{16}{y} > \frac{16}{0.1} = 160 > 24$$

(B) $\begin{cases} x \ge 0.1 \\ y \ge 1000 \end{cases}$ or $\begin{cases} y \ge 0.1 \\ x \ge 1000 \end{cases}$

$$C(x,y) = 2xy + \frac{16}{x} + \frac{16}{y} > 2xy \ge 2(0.1)\cdot(1000)$$
$$= 200 > 24$$

<u>Step 3</u>

- R is closed and bounded, C is continuous

Hence by EVT, $C|_R$ has minimum.

Critical point of C : $(2,2) \in \mathrm{int}(R)$

$C(2,2) = 24$

- $C(x,y) > 24$ on ∂R or R^c

$\therefore$ C has min value 24 at $(2,2)$ on $\hat{\Omega} = \{x>0, y>0\}$

size $2\times 2\times 4$.

Taylor series expansion.

<u>Recall</u> Taylor expansion for 1-variable function $g(t)$ at $t=0$ up to order k;

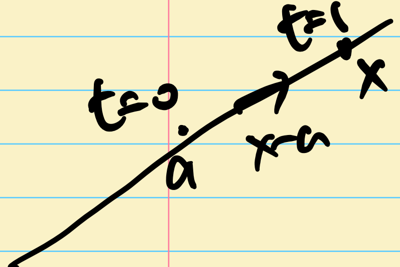
$$g(t) = g(0) + g'(0)t + \frac{1}{2!}g''(0)t^2 + \frac{1}{3!}g^{(3)}(0)t^3 + \cdots + \frac{1}{k!}g^{(k)}(0)t^k + (\text{remainder}) \qquad (*)$$

want: similar formula for a multi-variable function defined near $a \in \mathbb{R}^n$.

$x = (x_1 \text{ — } x_n)$, $a = (a_1 \text{ — } a_n)$

$f(x)$

Let $g(t) = f(a + t(x-a))$



If $\|x-a\|$ is small, then for $|t|\le 1$, $\|t(x-a)\| = |t|\,\|x-a\|$ is small and $g(t)$ is defined.

By (*), $f(a+t(x-a)) = g(0) + g'(0)t + \cdots + \frac{1}{k!} g^{(k)}(0)t^k$

+ remainder

Put $t=1$, $f(x) =$

Express $g^{(k)}(0)$ in terms of f.

$g(0) = f(a + 0\cdot(x-a)) = f(a)$

$$g'(t) = \frac{d}{dt} f(a+t(x-a))$$

$$= \nabla f(a+t(x-a)) \cdot (x-a)$$

$$= \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a+t(x-a))(x_i - a_i)$$

Put $t=0$, $g'(0) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a)(x_i - a_i)$

$$g''(t) = \frac{d}{dt} g'(t)$$

$$= \sum_{i=1}^{n} \frac{d}{dt}\left(\frac{\partial f}{\partial x_i}(a+t(x-a))(x_i - a_i)\right)$$

$$= \sum_{i=1}^{n}\left(\sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_j \partial x_i}(a+t(x-a))(x_j - a_j)\right)(x_i - a_i)$$

$$g''(0) = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_j \partial x_i}(a)\,(x_i - a_j)(x_i - a_i)$$

Hence, Taylor expansion at a up to order 2 is

$$f(x) = f(a) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a)(x_i - a_i)$$

$$+ \frac{1}{2!} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_j \partial x_i}(a)\,(x_i - a_i)(x_j - a_j)$$

+remainder.

eg If $n=2$, $f = f(x,y)$, $a = (x_0, y_0)$ and f is C^2 (so $f_{xy} = f_{yx}$) then

$$f(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

$$+ \frac{1}{2}\Big(f_{xx}(x_0,y_0)(x-x_0)^2 + 2 f_{xy}(x_0,y_0)(x-x_0)(y-y_0) + f_{yy}(x_0,y_0)(y-y_0)^2 \Big)$$

+ remainder.

Similarly, the general term is

$$g^{(k)}(0) = \sum_{i_1,\cdots,i_k=1}^{n} \frac{\partial^k f}{\partial x_{i_1} \cdots \partial x_{i_k}}(a)(x_{i_1}-a_{i_1})\cdots(x_{i_k}-a_{i_k})$$

Thm (Taylor's theorem)

$\Omega \subseteq \mathbb{R}^n$, $f: \Omega \to \mathbb{R}$ C^k function.

Then for any $x, a \in \Omega$,

$$f(x) = f(a) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a)(x_i - a_i)$$

$$+ \frac{1}{2!} \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(a)(x_i - a_i)(x_j - a_j)$$

$$+ \cdots$$

$$+ \frac{1}{k!} \sum_{i_1,\cdots,i_k=1}^{n} \frac{\partial^k f}{\partial x_{i_1} \cdots \partial x_{i_k}}(a)(x_{i_1}-a_{i_1})\cdots(x_{i_k}-a_{i_k})$$

$$+ \varepsilon_k(x, a)$$

with $\displaystyle \lim_{x \to a} \frac{\varepsilon_k(x,a)}{\|x-a\|^k} = 0.$

Def $P_k(x) \doteq f(a) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a)(x_i - a_i)$

$$+ \frac{1}{2!} \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(a)(x_i - a_i)(x_j - a_j)$$

$$+ \cdots$$

$$+ \frac{1}{k!} \sum_{i_1, \dots, i_k = 1}^{n} \frac{\partial^k f}{\partial x_{i_1} \cdots \partial x_{i_k}}(a)(x_{i_1} - a_{i_1}) \cdots (x_{i_k} - a_{i_k})$$

is called the <u>k-th order Taylor polynomial</u> of f at a.

Rmk ① $P_1(x) = f(a) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a)(x_i - a_i)$

$= L(x)$ linearization of f at a.

② P_k and f have equal partial derivatives up to order k at a.